Partial Orders and Topologies on Finite Sets

Demetres Christofides George Raptis

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A quasi-ordered set $(X, \prec)$ is a set X, together with a binary relation $\prec$ which is reflexive and transitive. The relation $\sim$ on X defined by $x \sim y$ if $x \prec y$ and $y \prec x$ is an equivalence relation on X. We can define a relation $\leq$ on $X/\sim = \{[x] : x \in X\}$ by $[x] \leq [y]$ if $x \prec y$. This is a well-defined quasi-order which is also antisymmetric. A quasi-ordered set X with a relation $\leq$ which is also antisymmetric is called a partially ordered set (poset). For $x, y \in X$ we write $x < y$ meaning $x \leq y$ and $x \neq y$. We say that $x, y \in X$ are incomparable if neither $x \leq y$ nor $y \leq x$.

The number of quasi-orders on a set of n elements is denoted by Q_n. The number of partial orders on a set of n elements is denoted by P_n. There are no explicit formulae for P_n and Q_n and the problem of finding such formulae is open and possibly very hard (See Table below).

n	P_n	Q_n
1	1	1
2	3	4
3	19	29
4	219	355
5	4231	6942
6	130023	209527

We will now mention some of the existing results concerning those sequences.

It is easy to see that there are exactly $3^{\binom{n}{2}}$ reflexive and antisymmetric relations on a set X of n elements. (For every two distinct $x, y \in X$, either $x \leq y$, or $y \leq x$ or x and y are incomparable). In particular,

$$P_n \leq 3^{\binom{n}{2}}.$$

Partitioning X into k non-empty subsets $X_1, \ldots, X_k$ and partially ordering the set of these subsets, we get a quasi-order on X. Clearly we can get every quasi-order on X uniquely in this way. If we denote by $S(n, k)$ the number of ways to partition a set of n elements into k non-empty subsets (this is known as the Stirling number of the second kind), we get that

$$Q_n = \sum_{k=1}^{n} S(n, k) P_k.$$

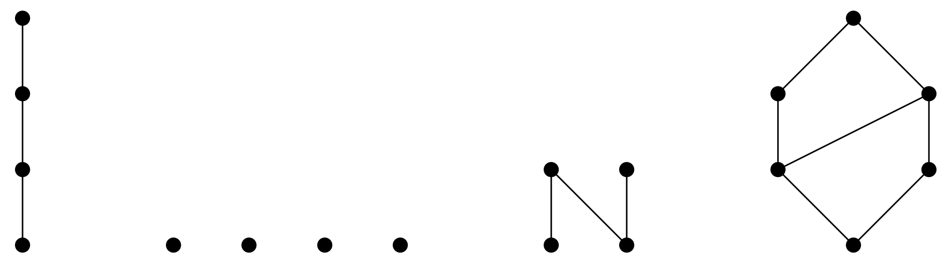
It is also known that $\lim_{n\to\infty} \frac{P_n}{Q_n} = 1$. For this, the reader may consult [3].

Let $(X, \leq)$ be a finite poset. We say that an element $x \in X$ covers $y \in X$ if $y < x$ and there does not exist $z \in X$ with $y < z$ and $z < x$. We can represent X with a diagram, by representing each element $x \in X$ by a distinct point, so that

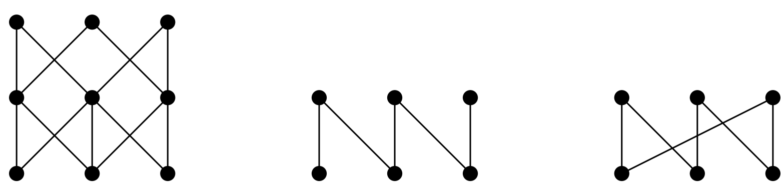
(i) Whenever $x < y$, the point representing y is higher than the point representing x. (Imagine this on the plane).

(ii) Whenever y covers x, the points representing the two elements are joined by a straight line segment.

This diagram is called the Hasse diagram of the poset X. (See Figures 1-7).



(1) Chain (2) Antichain (3) N-poset (4)



(5) (6) Fence (7) Crown

We can view the Hasse diagram of a poset as a directed graph. (There is a directed edge from x to y if y covers x). Clearly, every such directed graph cannot have either a directed cycle (by transitivity of $\leq$) or a cycle with all but one edge directed cyclically (by construction). It turns out that these two conditions are also sufficient for a directed graph to represent poset. (Interpreting each directed edge $\overrightarrow{xy}$ as 'y covers x'). In particular such a directed graph can have no triangles. Every bipartite graph on n vertices, with the edges directed from the one partite set to the other satisfies the above conditions. By considering only those with $\lfloor \frac{n}{2} \rfloor$ vertices in the one partite set, and $\lceil \frac{n}{2} \rceil$ in the other, we deduce that there are at least $2^{[\frac{n^2}{4}]}$ such directed graphs, and so

$$P_n \geq 2^{[\frac{n^2}{4}]}.$$

In fact, Kleitman and Rothschild in [5] proved that there is a constant C such that

$$\log_2 P_n \leq \frac{n^2}{4} + Cn^{\frac{3}{2}} \log n$$

for all n. Hence they deduced that

$$\log_2 P_n \sim \frac{n^2}{4}.$$

Now we will see how the problem of enumerating partial orders on a set X of n elements, turns out to be equivalent to the problem of enumerating topologies on X with a certain property. We give first some definitions:

A topological space $(X, \mathcal{T})$ is said to be a T_0-space if for every distinct $x, y \in X$ there is an open set U containing exactly one of them.

A topological space $(X, \mathcal{T})$ is said to be a T_1-space, if for every $x \in X$, $\{x\}$ is closed.

A topological space $(X, \mathcal{T})$ is said to be an A-space if any intersection of open sets is open. In particular any topology on a finite set is an A-topology.

Lemma 1. *Let X be a topological space.*

(i) If X is T_1 then X is also T_0.

(ii) If X is an A-space, then the only T_1 topology on X is the discrete topology.

(iii) If X is a finite T_0 space, then there is an $x \in X$ such that $\{x\}$ is closed.

Proof.

(i) Follows from the observation that $X - \{x\}$ is open.

(ii) Since an arbitrary union of closed sets is closed.

(iii) By induction. □

Theorem 2. *Let X be a finite set with n elements. Then*

(i) There is a 1-1 correspondence between the set of quasi-orders on X and the set of topologies on X.

(ii) There is a 1-1 correspondence between the set of partial orders on X and the set of T_0-topologies on X.

Proof.

(i) Given a quasi-order $\prec$ on X, we define a topology on X which has basis of open sets the sets of the form $U_x = \{y \in X : y \prec x\}$ for $x \in X$. This is indeed a basis since $U_x \cap U_y = \bigcup \{U_z : z \prec x \text{ and } z \prec y\}$. Conversely, given a topology $\mathcal{T}$ on X we can define a quasi-order $\prec$ on X by $x \prec y$ if and only if $V_x \subset V_y$ where $V_x = \bigcap \{U \in \mathcal{T} : x \in U\}$. It is clear that $\prec$ is a quasi-order on X, and the reader can check that these maps are mutual inverses.

(ii) Using the above correspondence, it suffices to show that a partial order corresponds to a T_0-space and vice-versa. Given a partial order $\leq$ on X, and distinct $x, y \in X$, then either x and y are incomparable in which case $x \in U_x$ but $y \notin U_x$, or they are comparable, say (without any loss of generality) $x \leq y$ in which case $x \in U_x$ but $y \notin U_x$. Conversely, given a T_0-topology on X, and two elements $x, y \in X$ with $x \prec y$ and $y \prec x$ then y belongs to every open set containing x and x belongs to every open set containing y. Therefore $x = y$, hence $\prec$ is a partial order. □

Remarks.

(1) The above proof is not valid for infinite sets because the two maps in (i) might no longer be inverses to each other, since V_x is not open in general.

(2) For an infinite set X, the above correspondence holds if we consider only A-topologies on X. The proof is essentially the same as above and is therefore left to the reader.

(3) In connection to this theorem, Lemma 1 says that the only poset (finite or infinite) corresponding to a T_1-topology is the antichain and that every finite poset has a maximal element.

(4) In [9] it is shown that on a set X of n elements, no topology other than the discrete has more than $\frac{3}{4}2^n$ open sets. This result is best possible, i.e. for $n \geq 2$ there is a topology on X with exactly $\frac{3}{4}2^n$ open sets. In [10], Stanley proved a stronger result.

Given a relation R on a set $X = \{x_1, \ldots, x_n\}$, we can define an $n \times n$ $(0,1)$-matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i R x_j, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a 1-1 correspondence between relations on X and $n \times n$ $(0,1)$-matrices. (Since R is a subset of $X \times X$). In the case where R is a quasi-order, the corresponding matrix A satisfies:

(i) $a_{ii} = 1$ for every i,

(ii) If $a_{ij} = a_{jk} = 1$, then $a_{ik} = 1$.

Conversely, we leave the reader to check that these conditions on A determine uniquely a quasi-order on X. In the case where R is a partial order, then A also has to satisfy:

(iii) For all i, j, $a_{ij} = a_{ji} = 1$ if and only if $i = j$.

Hence, by Theorem 2 there is a 1-1 correspondence between finite topological (T_0)-spaces on X, and $(0,1)$-matrices satisfying conditions (i),(ii) (and condition (iii)) above. This correspondence can be described in terms of the topology directly by:

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in \overline{\{x_i\}}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *A 0,1 $n \times n$ matrix A corresponds to a topological space (a quasi-order) on a set X of n elements, if and only if it has 1's on the main diagonal and satisfies $A^2 = A$ under Boolean operations*[1].

Proof. Such a matrix A clearly satisfies (i). If $a_{ij} = a_{jk} = 1$ for some i, j, k, then $a_{ik} = \sum_{j=1}^{n} a_{ij}a_{jk} = 1$, hence A also satisfies (ii). Conversely if a matrix A satisfies (i) and (ii), then we need to show that $a_{ik} = \sum_{j=1}^{n} a_{ij}a_{jk}$. If $a_{ik} = 0$, then we cannot have $a_{ij} = a_{jk} = 1$ for any j (by condition (ii)). If $a_{ik} = 1$, then $a_{ii}a_{ik} = 1$, so $\sum_{j=1}^{n} a_{ij}a_{jk} = 1$. □

A map f between two posets X, X' is called order preserving if for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$. (Where $\leq$ denotes the partial order in both X and X'). A poset X is said to have the fixed point property if every order preserving map $f : X \to X$ has a fixed point, i.e. there is an $x \in X$ with $f(x) = x$. The poset shown in Figure 5 has this property. The problem of characterizing the posets with the fixed point property is open. This has been answered in the positive (Knaster-Tarski theorem) in the case of complete posets[2]. (See Figure 4 for the Hasse diagram of a complete poset).

In the case of lattices[3], the above problem has been completely solved. Tarski in [11] proved that every complete lattice[4] has the fixed point property and Davis in [2] proved that every lattice with the fixed point property is complete.

It is now reasonable to ask what the relationship between the maps that preserve the order-structure and the maps that preserve the corresponding topological structure is. We will end our introduction to posets by completing the correspondence between partial orders and T_0-topological spaces with the following lemma:

Lemma 4. *A map f from a poset X into X is order preserving if and only if it is continuous with respect to the corresponding topology on X (as given in the proof of Theorem 2).*

Proof. If f is order preserving, then $f^{-1}(U_x) = \bigcup \{U_z : f(z) \leq x\}$ is open. Therefore f is continuous. Conversely, if f is continuous and $x, y \in X$ with $x \leq y$, then $U_x \subset U_y$. Since f is continuous, $f^{-1}(U_{f(y)})$ is an open set containing y, hence it also contains U_y and so it also contains U_x. It follows that $f(x) \in U_{f(y)}$, i.e $f(x) \leq f(y)$. Therefore f is order preserving. □

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[1] I.e. $1+1 = 1+0 = 0+1 = 1 \times 1 = 1, 0+0 = 0 \times 0 = 0 \times 1 = 1 \times 0 = 0$.

[2] A poset X is complete if for every subset Y of X there is an element $x \in X$ with $x \geq y$ for all $y \in Y$ and for all $z \in X$ with this property, $z \geq x$. We call x the least upper bound of Y.

[3] A lattice X is a poset with the additional property that every two distinct $x, y \in X$ have a least upper bound and a greatest lower bound, the latter defined in the obvious way. For example, the power set of a set ordered under inclusion.

[4] A lattice X is complete if every subset of X has a least upper bound and a greatest lower bound. In particular, every finite lattice is complete. See Figure 4 again.

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