# Pair Lengths of Product Graphs

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**Abstract**

The pair length of a graph $G$ is the maximum positive integer $k$, such that the vertex set of $G$ can be partitioned into disjoint pairs $\{x, x'\}$, such that $d(x, x') \geqslant k$ for every $x \in V(G)$ and $x'y'$ is an edge of $G$ whenever $xy$ is an edge. Chen asked whether the pair length of the cartesian product of two graphs is equal to the sum of their pair lengths. Our aim in this short note is to prove this result.

## 1 Introduction

Recently, Chen [1], in order to generalize a theorem of Graham, Entringer and Székely [2], introduced (for connected graphs) the following definition:

**Definition 1.** Let $k$ be a non-negative integer. A graph $G = (V, E)$ is ***k-pairable*** if $V(G)$ can be partitioned into disjoint sets $\{x, x'\}$ such that $d(x, x') \geqslant k$ for every $x \in V(G)$, and $xy \in E(G) \Rightarrow x'y' \in E(G)$. We call such a partition a ***k-pairing*** (or just a ***pairing***) of $G$.

In this paper we will be mainly concerned with connected graphs. We will make some remarks about disconnected graphs at the end of the paper.

It follows by the definition, that if $G$ is $k$-pairable, then it is also $l$-pairable for every $1 \leqslant l \leqslant k$. In the definition, we allow $x$ to equal $x'$. With this convention, every graph is 0-pairable. Note however that there are graphs which are not $k$-pairable for any positive integer $k$. For example, graphs with an odd number of vertices cannot have any partition into disjoint pairs. On the other hand, a disconnected graph might be $k$-pairable for every non-negative integer $k$. For example, consider the graph $G = 2K_2$, made up by two disjoint copies of $K_2$. These observations motivate the following definition in [1]:

**Definition 2.** The ***pair length*** $p(G)$ of a graph $G$ is the maximum $k$ such that $G$ is $k$-pairable. If $G$ is $k$-pairable for every $k$, then $p(G) = \infty$.

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Recall that the cartesian product of two graphs $G$ and $H$ is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(x_1, y_1)$ is adjacent to $(x_2, y_2)$ if and only if either $x_1 = x_2$ and $y_1$ is adjacent to $y_2$ in $H$, or $y_1 = y_2$ and $x_1$ is adjacent to $x_2$ in $G$.

Chen [1] showed that if $G$ is $k$-pairable and $H$ is $l$-pairable, then $G \times H$ is $(k+l)$-pairable. Indeed, given a $k$-pairing of $G$ into pairs $\{x, x'\}$, and an $l$-pairing of $H$ into pairs $\{y, y'\}$, it can be easily checked (using the property that $d((x,y),(x',y')) = d(x,x') + d(y,y')$), that the partition of $V(G \times H)$ into pairs $\{(x,y),(x',y')\}$, is a $(k+l)$-pairing of $G \times H$. This shows that $p(G \times H) \geqslant p(G) + p(H)$. Chen [1] asked whether equality always holds. We proceed in the next section to prove this result.

# 2 The main result

Note that a $k$-pairing of a graph $G$ defines an automorphism $f$ of $G$ of order at most 2, given by $f : V(G) \to V(G); x \mapsto x'$. Conversely, any automorphism $f$ of order at most 2 of a graph $G$ defines a $k$-pairing of $G$, for any non-negative integer $k \leqslant \min\{d(x, f(x)) : x \in V(G)\}$, by pairing $x$ with $f(x)$.

It is often simpler to think in terms of automorphisms rather than pairings. To introduce our ideas, suppose first that $\mathrm{Aut}(G \times H) = \mathrm{Aut}(G) \times \mathrm{Aut}(H)$ holds. Any pairing of $G \times H$ defines an automorphism $\phi$ of $G \times H$, of order at most 2, which must be of the form $(x,y) \mapsto (f(x), g(y))$ where $f$ and $g$ are automorphisms of $G$ and $H$ respectively. Since $\phi$ is of order at most 2, we deduce that $f$ and $g$ are also of order at most 2, so they define pairings of $G$ and $H$. Thus, arguing as in the end of Section 1, we deduce that $p(G \times H) = p(G) + p(H)$.

However, the automorphism group of $G \times H$, might be larger than $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$. For example, the automorphism group of $K_2$ is the cyclic group of order 2, while the automorphism group of the square $C_4 = K_2 \times K_2$, is the dihedral group of order 8. It might help the reader to note that this group is generated by the automorphism groups of the two copies of $K_2$, and the transposition between these two copies.

So what happens if $\mathrm{Aut}(G \times H)$ is larger than $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$? Fortunately, we can say quite a lot about $\mathrm{Aut}(G \times H)$. In some sense, all the 'extra' automorphisms of $\mathrm{Aut}(G \times H)$ are generated in the same way as in the above example. We cannot define 'such' automorphisms directly, but it turns out that if we go down to the level of indecomposable graphs then we can.

Call a connected graph $G$ ***indecomposable***, if $G = G_1 \times G_2$ implies that either $G_1$ or $G_2$ is the trivial graph $K_1$. Sabidussi [4] in 1960 and independently Vizing [5] in 1963 proved the following:

**Theorem 3.** *Every connected graph can be written uniquely as a cartesian product of indecomposable factors.*

It is known that this theorem does not hold for a disconnected graph. For example, see [3, Theorem 4.2].

Sabidussi [4] also proved the following:

**Theorem 4.** *The automorphism group of a connected graph $G$ is generated by the automorphism groups of its indecomposable factors and by transpositions between isomorphic indecomposable factors.*

The appearance of a fixed point (or similar) is crucial in the proof of our main result. We first consider a special case which contains the idea.

**Lemma 5.** *Let $G$ be a connected indecomposable graph. Then $p(G \times G) = 2p(G)$.*

*Proof.* It is enough to show that $G \times G$ has no $k$-pairings, for any $k > 2p(G)$. So fix a pairing of $G \times G$, and let $\phi$ be the corresponding automorphism of $G$. By Theorem 4, there are automorphisms $f, g$ of $G$ such either $(x, y) \mapsto (f(x), g(y))$ for every $x, y \in V(G)$, or $(x, y) \mapsto (f(y), g(x))$ for every $x, y \in V(G)$. In the first case, since $\phi$ is of order at most 2, we deduce that both $f$ and $g$ have order at most 2. Therefore, they define pairings of $G$, and so we can choose $x, y \in V(G)$ such that $d(x, f(x))$ and $d(y, g(y))$ are at most $p(G)$. But then the distance in $G \times G$ between $(x, y)$ and $(f(x), g(y))$ is at most $2p(G)$ as required. In the second case, the fact that $\phi$ is of order at most 2 shows that $fg(x) = x$ and $gf(y) = y$ for every $x, y \in V(G)$. It follows that $g = f^{-1}$. But then it is easily seen that $(x, f^{-1}(x))$ is a fixed point of $\phi$ and this completes the proof as the distance between this point and its image is 0 which is certainly at most $2p(G)$. □

We are now ready to prove our main result. The proof is no more difficult than the proof of the previous lemma.

**Theorem 6.** *Let $G$ be a connected graph and suppose that $G = G_1 \times \ldots \times G_n$ as a product of indecomposable factors. Then $p(G) = \sum_{i=1}^{n} p(G_i)$.*

*Proof.* Write $G$ as $X_1^{n_1} \times \ldots \times X_k^{n_k}$ where $X_1, \ldots, X_k$ are indecomposable and pairwise non-isomorphic. By Theorem 4, $\mathrm{Aut}(G) = \mathrm{Aut}(X_1^{n_1}) \times \ldots \times \mathrm{Aut}(X_k^{n_k})$ and so by what we have shown so far, $p(G) = \sum_{i=1}^{k} p(X_i^{n_i})$. Therefore, it is enough to consider the case that $G = X^n$ for some indecomposable graph $X$. We may assume that $X \neq K_1$. Fix a pairing of $G$, and let $\phi$ be the corresponding automorphism. By Theorem 4, $\phi$ is of the form

$$\phi : (x_1, \ldots, x_n) \mapsto (f_1(x_{\sigma(1)}), \ldots, f_n(x_{\sigma(n)})),$$

for some automorphisms $f_1, \ldots, f_n$ of $X$, and a permutation $\sigma$ of $[n]$. But $\phi$ has order at most 2, so

$$(x_1, \ldots, x_i, \ldots, x_n) = (f_1 f_{\sigma(1)}(x_{\sigma^2(1)}), \ldots, f_i f_{\sigma(i)}(x_{\sigma^2(i)}), \ldots, f_n f_{\sigma(n)}(x_{\sigma^2(n)}))$$

for every $x_1, \ldots, x_n \in V(X)$. It follows that $\sigma$ has order at most 2. Indeed, if for example $\sigma^2(1) = i \neq 1$ then we can just pick any $x_i \in V(X)$ such that $x_i \neq f_{\sigma(1)}^{-1} f_1^{-1}(x_1)$ to get a contradiction. Hence, the disjoint cycle decomposition of $\sigma$ consists only of transpositions (and 1-cycles). Moreover, if $(ij)$ is in the cycle decomposition of $\sigma$, (whether $i = j$ or not,) then $f_i f_j(x_i) = x_i$ and $f_j f_i(x_j) = x_j$ for every $x_i, x_j \in X$. It follows that $f_j = f_i^{-1}$. Suppose, without loss of generality, that $\sigma = (12)...((2r-1)(2r))(2r+1)...(n)$. For each $2r+1 \leqslant i \leqslant n$, $f_i$ has order at most 2, so we can choose an $x_i \in V(X)$ such that $d(x_i, f_i(x_i)) \leqslant p(X)$. Consider now the point

$$(x, f_1^{-1}(x), \ldots, x, f_{2r-1}^{-1}(x), x_{2r+1}, \ldots, x_n),$$

where $x$ is any element of $V(X)$. It is mapped to

$$(x, f_1^{-1}(x), \ldots, x, f_{2r-1}^{-1}(x), f_{2r+1}(x_{2r+1}), \ldots f_n(x_n))$$

under $\phi$. But the distance of these two points in $G$ is at most $(n-2r)p(X) \leqslant np(X)$. This proves the result. □

**Corollary 7.** *Let $G, H$ be connected graphs. Then $p(G \times H) = p(G) + p(H)$.* □

Finally, let us turn our attention to disconnected graphs. Let $G$ be a disconnected graph and let $f$ be an automorphism of $G$ of order at most 2. Pick $x \in V(G)$ and suppose that $x$ and $f(x)$ belong to different connected components of $G$. Then, as $f$ is an automorphism, we must have that the component $G_1$ of $G$ containing $x$ and the component $G_2$ of $G$ containing $f(x)$ are isomorphic, and moreover, $f$ restricted to $G_1$, defines an isomorphism between $G_1$ and $G_2$.

**Lemma 8.** *Let $G$ be a graph, and let $G_1, \ldots, G_n$ be its connected components. Suppose that $G_{2i-1} \cong G_{2i}$ for $1 \leqslant i \leqslant k$ and that $G_{2k+1}, \ldots, G_n$ are pairwise non-isomorphic. Then*

$$p(G) = \begin{cases} \min\{p(G_i) : 2k+1 \leqslant i \leqslant n\} & \text{if } n > 2k, \\ \infty & \text{otherwise.} \end{cases}$$

*Proof.* If $n = 2k$, then we can find an automorphism of order 2 of $G$ which interchanges $G_{2i-1}$ with $G_{2i}$ for every $1 \leqslant i \leqslant k$. Therefore, $p(G) = \infty$ in this case. On the other hand, if $n > 2k$ we may assume without loss of generality that $p(G_n) = \min\{p(G_i) : 2k+1 \leqslant i \leqslant n\}$. But there is an odd number of connected components of $G$ which are isomorphic to $G_n$, so by the above discussion, there must be an isomorphic copy $H$ of $G_n$ in $G$, such that $f$ restricted to $H$, is an automorphism of order at most 2 of $H$. It follows that $p(G) \leqslant p(H) = p(G_n)$. This settles the case $n > 2k$, as it is easy to construct a $p(G_n)$-pairing of $G$. □

**Theorem 9.** *Given graphs $G, H$, we have $p(G \times H) = p(G) + p(H)$.*

*Proof.* Suppose that $G_1, \ldots, G_n$ are the connected components of $G$ with $G_{2i-1} \cong G_{2i}$ for $1 \leqslant i \leqslant k$ and $G_{2k+1}, \ldots, G_n$ are pairwise non-isomorphic. Suppose also that $H_1, \ldots, H_{n'}$ are the connected components of $H$ with $H_{2j-1} \cong H_{2j}$ for $1 \leqslant j \leqslant k'$ and $H_{2k'+1}, \ldots, H_{n'}$ are pairwise non-isomorphic. We have that the connected components of $G \times H$ are $G_i \times H_j$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n'$. By the previous lemma we may assume that $k = k' = 0$. Without loss of generality we may assume $p(G_1) \leqslant p(H_1)$. Then either $H$ contains no connected component which is isomorphic to $G_1$, in which case $G \times H$ contains exactly one connected component isomorphic to $G_1 \times H_1$, or

$H$ contains a connected component isomorphic to $G_1$, in which case $G \times H$ contains exactly one connected component isomorphic to $G_1 \times G_1$. In both cases the result follows. $\square$

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