A COUNTEREXAMPLE TO A CONJECTURE OF SIMONOVITS AND SOS

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Abstract. We say that a family $\mathcal{G}$ of graphs is P_3-intersecting if any two graphs in the family intersect on a path of length 3. For each $n \geqslant 6$ we construct a family $\mathcal{G}_n$ of subgraphs of K_n which is P_3-intersecting and has size larger than $\frac{1}{8} \cdot 2^{\binom{n}{2}}$. This disproves a conjecture of Simonovits and Sós.

In extremal combinatorics, we are often interested in the maximal size of a combinatorial structure satisfying certain restrictions. The typical example of this is the Erdős-Ko-Rado theorem [4] which states that the maximal size of an intersecting family (i.e. any two sets in the family have non-empty intersection) of k-element subsets of an n-element set with $k < n/2$ is at most $\frac{k}{n}\binom{n}{k}$ and this maximum is achieved only when we take all k-element subsets containing a fixed element of the ground set.

The Erdős-Ko-Rado theorem has been generalised in numerous directions. One such direction, proposed by Simonovits and Sós (see [2]) is the following: Let F be a fixed graph. We say that a family $\mathcal{G}$ of graphs is F-intersecting if any two graphs in $\mathcal{G}$ contain an isomorphic copy of F in their intersection. The obvious generalisation of the Erdős-Ko-Rado theorem is to ask about the maximal size of an F-intersecting family of subgraphs of K_n. The natural conjecture would be that the best we can do is to fix a specific copy of F in K_n and take all subgraphs of K_n containing this particular copy. This family has size $2^{-e(F)}2^{\binom{n}{2}}$, where $e(F)$ denotes the number of edges of F.

Given a graph G and a positive integer n, let us denote by $c_n = c_n(G)$ the minimum non-negative real number such that every G-intersecting family of subgraphs of K_n has size at most $c_n 2^{\binom{n}{2}}$. It is not difficult to see that the sequence $c_n(G)$ is decreasing and therefore tends to a limit which we denote by $c(G)$. The aim would be to determine $c(G)$ for every graph G but at the moment very little is known about the problem. It is trivial to see that for every graph G, we have $2^{-e(G)} \leqslant c(G) \leqslant 1/2$. It is known that $c(G) = 1/2$ whenever G is a star-forest and it is conjectured (see e.g. [1]) that $c(G) < 1/2$ otherwise. (This is known to be true for non-bipartite graphs.) Simonovits and Sós (see [2]) conjectured that $c(P_3) = c(K_3) = 1/8$. Very recently, an important breakthrough has been achieved by Ellis, Filmus and Friedgut [3] who proved the Simonovits-Sós conjecture for K_3 showing that $c(K_3) = 1/8$. (In fact they proved a much stronger result.)

However, nothing more was known about P_3 and in this short note we will prove that $c(P_3) \geqslant 17/128 > 1/8$ thus providing a counterexample to the Simonovits-Sós conjecture for P_3. Observe that it is enough to exhibit a graph G and a P_3-intersecting family $\mathcal{G}$ of subgraphs of G of size $\frac{17}{128} \cdot 2^{e(G)}$. Indeed if we can construct such a family $\mathcal{G}$, then given any graph H containing G, the family $\mathcal{H}$ of all subgraphs of H containing a member of $\mathcal{G}$ is P_3-intersecting of size $\frac{17}{128} \cdot 2^{e(H)}$.

We will construct such an example with G being a graph on six vertices and seven edges. Before giving this counterexample, we begin with a 'near counterexample'. Let G be the complete graph on vertex sets V_1, V_2 where $|V_1| = 2$ and $|V_2| = 3$. Let $\mathcal{A}$ be the family of all subgraphs of G which have at least five edges and let $\mathcal{B}$ be the family of all subgraphs of G which are isomorphic to a four-cycle. It is immediate that $|\mathcal{A}| = 7$ and $|\mathcal{B}| = 3$. Moreover, it is very easy to check that any $G_1, G_2 \in \mathcal{A} \cup \mathcal{B}$ intersect in a path of length three unless

$G_1, G_2 \in \mathcal{B}$, in which case $G_1 \cap G_2$ intersect in a path of length 2 with the special property that both of its end-points belong to V_1. Now let G' be a new graph obtained from G by adding one new vertex, say x, and one new edge, say e, which is incident to x and one of the two vertices of V_1. Let $\mathcal{A}' = \mathcal{A} \cup (\mathcal{A} + e)$ and $\mathcal{B}' = \mathcal{B} + e$, where the family $\mathcal{C} + e$ denotes the set of all graphs obtained from the graphs in $\mathcal{C}$ by adding to them (vertex x and) the edge e. By the previous observations the family $\mathcal{A}' \cup \mathcal{B}'$ is P_3-intersecting with size $2|\mathcal{A}| + |\mathcal{B}| = 17$. Since $e(G') = 7$ we obtain the promised counterexample.

At the moment we do not have any conjecture to propose for the value of $c(P_3)$. We believe that $c(P_3) < 1/2$ but unfortunately we cannot prove it.

References

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