A note on hitting maximum and maximal cliques with a stable set

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Abstract

It was recently proved that any graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ contains a stable set hitting every maximum clique. In this note we prove that the same is true for graphs satisfying $\omega \geq \frac{2}{3}(\Delta+1)$ unless the graph is the strong product of $K_{\omega/2}$ and an odd hole. We also provide a counterexample to a recent conjecture on the existence of a stable set hitting every sufficiently large maximal clique.

1 Introduction

Given two graphs G and H, the *strong product* of G and H, denoted by $G \boxtimes H$, is the graph obtained by substituting each vertex in G with a copy of H. The graph $C_5 \boxtimes K_3$ (see Figure 1) has appeared as an exemplary graph in several situations, including as a counterexample to Hajós' conjecture [3] and as proof of tightness of the Borodin-Kostochka conjecture [2], Reed's ω, Δ, χ conjecture [10], and most recently a result on hitting all maximum cliques with a stable set:

Theorem 1 (King [7]). *Any graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ contains a stable set that intersects every maximum clique.*

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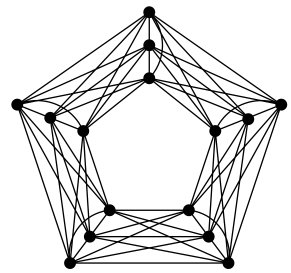

Figure 1: $C_5 \boxtimes K_3$

Since $C_5 \boxtimes K_3$ satisfies $\omega = \frac{2}{3}(\Delta + 1)$ but contains no stable set hitting every maximum clique, the strict inequality is necessary. Actually C_5 itself also shows that strictness is necessary, and is not just a Brooks-type exception. In the next two sections of this note we prove that any graph that exhibits this property is the strong product of a clique and an odd hole[1]:

Theorem 2. *Any connected graph satisfying $\omega \geq \frac{2}{3}(\Delta+1)$ contains a stable set intersecting every maximum clique unless it is the strong product of a clique and an odd hole.*

It is easy to confirm that the strong product of a clique and an odd hole does not contain a stable set hitting every maximum clique. In the last section of this note, we prove that there is no hope of proving a statement analogous to Theorem 1 for maximal rather than maximum cliques.

2 The clique graph

Following the approach of [7] and [9], we approach Theorem 2 by characterizing the structure of the *clique graph*. Given a graph G and a collection $\mathcal{C}$ of maximum cliques in G, we define the clique graph, denoted by $G(\mathcal{C})$, as follows. The vertices of $G(\mathcal{C})$ correspond to the cliques in $\mathcal{C}$; two vertices of $G(\mathcal{C})$ are adjacent if and only if their corresponding cliques intersect in G.

For now we can restrict our attention to connected clique graphs. When $\omega > \frac{2}{3}(\Delta + 1)$, we are guaranteed that if $G(\mathcal{C})$ is connected, then $|\cap \mathcal{C}| \geq \frac{1}{3}(\Delta + 1)$ [7]. However, the same is not necessarily true when $\omega = \frac{2}{3}(\Delta + 1)$, for example with the strong product of a clique and either a hole (i.e. a cycle of length ≥ 4) or P_ℓ (i.e. a path on ℓ vertices) for $\ell \geq 4$, in which case $\cap\mathcal{C}$ is empty. This is actually the only troublesome case. To prove this we need Hajnal's set collection lemma.

Lemma 3 (Hajnal [5])**.** *Let G be a graph and let $\mathcal{C}$ be a collection of maximum cliques in G. Then*

$$|\cap \mathcal{C}| + |\cup \mathcal{C}| \geq 2\omega(G).$$

[1] A *hole* is an induced cycle of length at least 4.

The following lemma extends a lemma of Kostochka [8] that is instrumental to the proof of Theorem 1.

Lemma 4. *Suppose G is connected and satisfies $\omega \geq \frac{2}{3}(\Delta+1)$, and let $\mathcal{C}$ be a collection of maximum cliques in G such that $G(\mathcal{C})$ is connected. Then either $|\cap\mathcal{C}| \geq \frac{1}{3}(\Delta+1)$, or G contains as a subgraph the strong product of $K_{\omega/2}$ and a connected graph of maximum degree 2 on at least four vertices. Furthermore, if G contains as a subgraph the strong product of a clique and a hole, then this subgraph is G itself.*

Proof. Suppose that $\mathcal{C}$ is a collection of maximum cliques such that $|\cap\mathcal{C}| < \frac{1}{3}(\Delta+1)$ and such that $G(\mathcal{C})$ is connected. Let $\mathcal{C}' \subset \mathcal{C}$ be a maximal collection of cliques with $|\cap\mathcal{C}'| \geq \frac{1}{3}(\Delta+1)$. Observe that any pair of intersecting maximum cliques must have at least $\frac{1}{3}(\Delta+1)$ vertices in their intersection, so $|\mathcal{C}'| \geq 2$.

Let C' and C'' be two non-disjoint cliques in $\mathcal{C}$ such that $C' \in \mathcal{C}'$ and $C'' \notin \mathcal{C}'$. Let $\mathcal{C}''$ denote $\mathcal{C}' \cup \{C''\}$. By the maximality of $\mathcal{C}'$, we have $|\cap\mathcal{C}''| < \frac{1}{3}(\Delta+1)$. Suppose that $\cap\mathcal{C}''$ is nonempty. Any vertex in $\cap\mathcal{C}''$ is adjacent to the rest of $\cup\mathcal{C}''$, so $|\cup\mathcal{C}''| \leq \Delta+1$. But this contradicts Lemma 3, so $\cap\mathcal{C}''$ must indeed be empty.

Since $C' \cap C'' \neq \emptyset$ it follows that $|C' \cap C''| \geq \omega/2$. On the other hand we also have $|C' \setminus C''| \geq |\cap\mathcal{C}'| \geq \omega/2$ and so $|C' \cap C''| = |\cap\mathcal{C}'| = \omega/2$. Thus it is clear that the sets $(C' \cap C'')$ and $(\cap\mathcal{C}')$ partition C'. Also, no clique of $\mathcal{C}'$ can intersect $C'' \setminus C'$, since a vertex in this intersection would be complete to C', contradicting the fact that C' is a maximum clique. Further, no clique C of $\mathcal{C}'$ other than C' can intersect C'', since this would imply that C and C'' have nonempty intersection of size less than $\frac{1}{3}(\Delta+1)$, which is impossible. Finally, it follows that there can be only one clique in $\mathcal{C}' \setminus \{C'\}$, otherwise $|\cup\mathcal{C}'|$ would be greater than $\Delta+1$.

We have shown that if $|\cap\mathcal{C}| < \frac{1}{3}(\Delta+1)$, then no three cliques of $\mathcal{C}$ intersect, and any two intersecting cliques intersect in exactly $\frac{1}{3}(\Delta+1)$ vertices. The first part of the lemma follows immediately from the fact that the strong product of a clique and a 3-cycle is itself a clique, thus $G[\mathcal{C}]$ is either a path or a hole, and if it is a path it must have at least four vertices, since $\cap\mathcal{C}$ is nonempty. The second part follows from the fact that if G contains the strong product of a clique and a hole as a subgraph G', then G' is Δ-regular and therefore since G is connected we have $G' = G$. $\square$

3 Hitting the maximum cliques with a stable set

In order to find our desired stable set, we need the main intermediate result in the proof of Theorem 1.

Theorem 5 (King [7])**.** *For a positive integer k, let G be a graph with vertices partitioned into cliques $V_1, \ldots V_r$. If for every i and every $v \in V_i$, v has at most $\min\{k, |V_i| - k\}$ neighbours outside V_i, then G contains a stable set of size r.*

Proof of Theorem 2. For fixed $\omega(G) \geq 1$ we proceed by induction on $|V(G)|$; the result trivially holds whenever $|V(G)| \leq \omega(G)$. Let $\mathcal{C}$ be the set of maximum cliques in a graph G, and

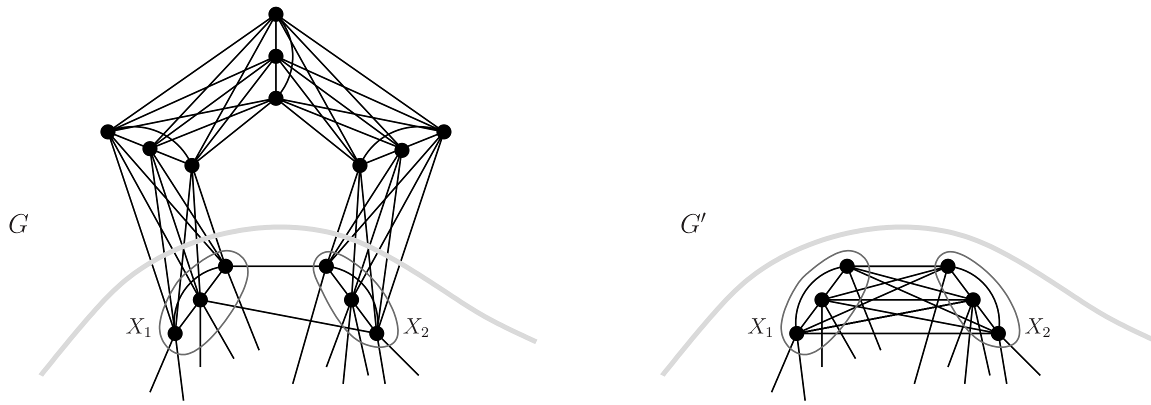

Figure 2: A reduction of a clique path for $\ell = 5$

let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ be the partitioning of $\mathcal{C}$ such that $G[\mathcal{C}_1], G[\mathcal{C}_2], \dots, G[\mathcal{C}_k]$ are the connected components of the clique graph $G[\mathcal{C}]$. We consider two cases. The first case is basically the same as the proof of Theorem 1.

Case 1: For every $1 \leq i \leq k$, $\cap\mathcal{C}_i \neq \emptyset$.

By Lemma 4, for every $1 \leq i \leq k$ we have $|\cap\mathcal{C}_i| \geq \frac{1}{3}(\Delta(G)+1)$. It suffices to show that there is a stable set in G intersecting each $\cap\mathcal{C}_i$. For a given i, every vertex in $\cap\mathcal{C}_i$ has at most $\Delta(G)+1-|\cup\mathcal{C}_i|$ neighbours in $\cup_{j\neq i}(\cap\mathcal{C}_j)$. Lemma 3 tells us that $|\cup\mathcal{C}_i|+|\cap\mathcal{C}_i| \geq \frac{4}{3}(\Delta(G)+1)$. Therefore $\Delta(G)+1-|\cup\mathcal{C}_i| \leq |\cap\mathcal{C}_i| - \frac{1}{3}(\Delta(G)+1)$. And since $|\cup\mathcal{C}_i| \geq \omega(G) \geq \frac{2}{3}(\Delta(G)+1)$, we can see that a vertex in $\cap\mathcal{C}_i$ has at most $\min\{\frac{1}{3}(\Delta(G)+1), |\cap\mathcal{C}_i| - \frac{1}{3}(\Delta(G)+1)\}$ neighbours in $\cup_{j\neq i}(\cap\mathcal{C}_j)$. It therefore follows from Theorem 5 that there is a stable set in G intersecting each $\cap\mathcal{C}_i$. This completes Case 1.

Case 2: For some $1 \leq i \leq k$, $\cap\mathcal{C}_i = \emptyset$.

Assume that $\cap\mathcal{C}_1 = \emptyset$. Lemma 4 tells us that either G is the strong product of $K_{\omega(G)/2}$ and a hole, or $G[\cup\mathcal{C}_i]$ contains as a subgraph the strong product of $K_{\omega(G)/2}$ and a P_ℓ for $\ell \geq 4$. In the former case the theorem clearly holds, so let us consider the latter case. If there is a vertex not in a clique of size $\omega(G)$, we can delete it and apply induction, so assume that no such vertex exists. Let the cliques of $\mathcal{C}_1$ be $C_1, \dots, C_{\ell-1}$ such that for $1 \leq i \leq \ell-2$, C_i and C_{i+1} intersect in exactly $\omega(G)/2$ vertices. Let X_1 denote $C_1 \setminus C_2$ and let X_2 denote $C_{\ell-1} \setminus C_{\ell-2}$.

We will construct a graph G' on fewer than $|V(G)|$ vertices such that $\omega(G') = \omega(G)$ and $\Delta(G') \leq \Delta(G)$, and apply induction to prove our result. To construct G' from G we delete $\cup_{1\leq i\leq \ell-2}(C_i \cap C_{i+1}) = (\cup\mathcal{C}_1) \setminus (X_1 \cup X_2)$ and add edges to make $X_1 \cup X_2$ a clique of size ω in G' (see Figure 2). Clearly G' has maximum degree at most $\Delta(G)$. We claim that G' has clique number $\omega(G)$. Suppose this is not the case. It follows that there exists a set $Y_1 \subseteq X_1 \cup X_2$ and a set Y_2 in $V(G) \setminus \cup\mathcal{C}_1$ such that $Y_1 \cup Y_2$ is a clique of size greater than $\omega(G)$. Let v be a vertex in Y_2. Since v is in an $\omega(G)$-clique in G, it has at most $\omega(G)/2$

neighbours in $X_1 \cup X_2$, so $|Y_1| \leq \omega(G)/2$. Therefore $|Y_2| > \omega(G)/2$, which implies that some vertex in Y_1 has at least $\omega(G) - 1$ neighbours in $\cup \mathcal{C}_1$ and more than $\omega(G)/2$ neighbours in Y_2, contradicting the fact that $\omega(G) \geq \frac{2}{3}(\Delta(G) + 1)$. Therefore G' has clique number $\omega(G)$.

By induction, there is a stable set S in G' hitting every $\omega(G)$-clique. Thus S is also a stable set in G intersecting $X_1 \cup X_2$ exactly once. Without loss of generality let v be a vertex in $X_1 \cap S$. From S we will construct a stable set S' hitting every $\omega(G)$-clique in G in one of two ways, depending on the parity of ℓ.

If ℓ is even, let S' consist of S along with one vertex in $C_{2k} \cap C_{2k+1}$ for each $1 \leq k \leq (\ell/2) - 1$. It is a routine exercise to confirm that S' is a stable set hitting every maximum clique in G.

If ℓ is odd, let S' consist of $S \setminus \{v\}$ along with $C_{2k-1} \cap C_{2k}$ for each $1 \leq k \leq (\ell - 1)/2$. Again we can easily confirm that S' is a stable set hitting every maximum clique in G, because the only $\omega(G)$-clique intersecting $C_1 \setminus C_2$ is C_1. This completes the proof. □

4 Hitting large maximal cliques with a stable set

Theorem 1 can be used to characterize minimum counterexamples to Reed's χ, ω, Δ conjecture; see for example [1] §4. Motivated by the problem of similarly characterizing minimum counterexamples to the local strengthening of Reed's χ, ω, Δ conjecture (see [4, 6]), King recently proposed the following unpublished conjecture:

Conjecture 6. *There exists a universal constant $\epsilon > 0$ such that every graph contains a stable set hitting every maximal clique of size at least $(1 - \epsilon)(\Delta + 1)$.*

We conclude this note by disproving the conjecture.

Theorem 7. *For any $\epsilon > 0$ there exists a graph in which every maximal clique has size at least $(1 - \epsilon)(\Delta + 1)$, and no stable set hits every maximal clique.*

Proof. Choose two positive integers k and t sufficiently large such that

$$(1 - \epsilon)(kt + 5t - 5) < kt + 2 - k. \tag{1}$$

We now construct a graph with vertices partitioned into sets A and B of size kt and $5t$ respectively. We further partition A into $A_1, \ldots, A_t$ and B into $B_1, \ldots, B_t$ such that

1. A is a clique and each A_i has size k
2. each B_i induces a 5-cycle, and there are no edges between B_i and B_j for $i \neq j$
3. vertices $u \in A_i$ and $v \in B_j$ are adjacent precisely when $i \neq j$.

Thus we can see that the unique maximum clique in G is $\cup_i A_i$, with size kt. All other maximal cliques of G consist of two vertices in B and $k(t-1)$ vertices of A. The maximum degree of the graph is $kt + 5t - 6$, achieved by all vertices in A. By (1), every maximal clique has size greater than $(1 - \epsilon)(\Delta + 1)$.

It therefore suffices to prove that no stable set intersects every maximal clique. Suppose we have a stable set S intersecting every maximal clique. Since A is a maximal clique, without loss of generality we can assume S intersects A_1, and therefore $S \setminus A_1 \subseteq B_1$. But then there must remain two adjacent vertices in $B_1 \setminus S$. Together with $\cup_{j \neq 1} A_j$ these vertices form a maximal clique in G. This contradiction completes the proof. $\square$

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