

# A note on hitting maximum and maximal cliques with a stable set

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## Abstract

It was recently proved that any graph satisfying  $\omega > \frac{2}{3}(\Delta + 1)$  contains a stable set hitting every maximum clique. In this note we prove that the same is true for graphs satisfying  $\omega \geq \frac{2}{3}(\Delta + 1)$  unless the graph is the strong product of  $K_{\omega/2}$  and an odd hole. We also provide a counterexample to a recent conjecture on the existence of a stable set hitting every sufficiently large maximal clique.

## 1 Introduction

Given two graphs  $G$  and  $H$ , the *strong product* of  $G$  and  $H$ , denoted by  $G \boxtimes H$ , is the graph obtained by substituting each vertex in  $G$  with a copy of  $H$ . The graph  $C_5 \boxtimes K_3$  (see Figure 1) has appeared as an exemplary graph in several situations, including as a counterexample to Hajós' conjecture [3] and as proof of tightness of the Borodin-Kostochka conjecture [2], Reed's  $\omega$ ,  $\Delta$ ,  $\chi$  conjecture [10], and most recently a result on hitting all maximum cliques with a stable set:

**Theorem 1** (King [7]). *Any graph satisfying  $\omega > \frac{2}{3}(\Delta + 1)$  contains a stable set that intersects every maximum clique.*

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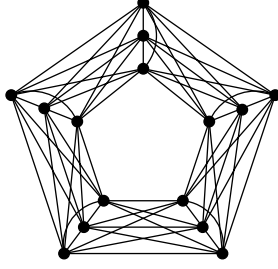


Figure 1:  $C_5 \boxtimes K_3$

Since  $C_5 \boxtimes K_3$  satisfies  $\omega = \frac{2}{3}(\Delta + 1)$  but contains no stable set hitting every maximum clique, the strict inequality is necessary. Actually  $C_5$  itself also shows that strictness is necessary, and is not just a Brooks-type exception. In the next two sections of this note we prove that any graph that exhibits this property is the strong product of a clique and an odd hole<sup>1</sup>:

**Theorem 2.** *Any connected graph satisfying  $\omega \geq \frac{2}{3}(\Delta + 1)$  contains a stable set intersecting every maximum clique unless it is the strong product of a clique and an odd hole.*

It is easy to confirm that the strong product of a clique and an odd hole does not contain a stable set hitting every maximum clique. In the last section of this note, we prove that there is no hope of proving a statement analogous to Theorem 1 for maximal rather than maximum cliques.

## 2 The clique graph

Following the approach of [7] and [9], we approach Theorem 2 by characterizing the structure of the *clique graph*. Given a graph  $G$  and a collection  $\mathcal{C}$  of maximum cliques in  $G$ , we define the clique graph, denoted by  $G(\mathcal{C})$ , as follows. The vertices of  $G(\mathcal{C})$  correspond to the cliques in  $\mathcal{C}$ ; two vertices of  $G(\mathcal{C})$  are adjacent if and only if their corresponding cliques intersect in  $G$ .

For now we can restrict our attention to connected clique graphs. When  $\omega > \frac{2}{3}(\Delta + 1)$ , we are guaranteed that if  $G(\mathcal{C})$  is connected, then  $|\cap \mathcal{C}| \geq \frac{1}{3}(\Delta + 1)$  [7]. However, the same is not necessarily true when  $\omega = \frac{2}{3}(\Delta + 1)$ , for example with the strong product of a clique and either a hole (i.e. a cycle of length  $\geq 4$ ) or  $P_\ell$  (i.e. a path on  $\ell$  vertices) for  $\ell \geq 4$ , in which case  $\cap \mathcal{C}$  is empty. This is actually the only troublesome case. To prove this we need Hajnal's set collection lemma.

**Lemma 3** (Hajnal [5]). *Let  $G$  be a graph and let  $\mathcal{C}$  be a collection of maximum cliques in  $G$ . Then*

$$|\cap \mathcal{C}| + |\cup \mathcal{C}| \geq 2\omega(G).$$

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<sup>1</sup>A *hole* is an induced cycle of length at least 4.

The following lemma extends a lemma of Kostochka [8] that is instrumental to the proof of Theorem 1.

**Lemma 4.** *Suppose  $G$  is connected and satisfies  $\omega \geq \frac{2}{3}(\Delta + 1)$ , and let  $\mathcal{C}$  be a collection of maximum cliques in  $G$  such that  $G(\mathcal{C})$  is connected. Then either  $|\cap \mathcal{C}| \geq \frac{1}{3}(\Delta + 1)$ , or  $G$  contains as a subgraph the strong product of  $K_{\omega/2}$  and a connected graph of maximum degree 2 on at least four vertices. Furthermore, if  $G$  contains as a subgraph the strong product of a clique and a hole, then this subgraph is  $G$  itself.*

*Proof.* Suppose that  $\mathcal{C}$  is a collection of maximum cliques such that  $|\cap \mathcal{C}| < \frac{1}{3}(\Delta + 1)$  and such that  $G(\mathcal{C})$  is connected. Let  $\mathcal{C}' \subset \mathcal{C}$  be a maximal collection of cliques with  $|\cap \mathcal{C}'| \geq \frac{1}{3}(\Delta + 1)$ . Observe that any pair of intersecting maximum cliques must have at least  $\frac{1}{3}(\Delta + 1)$  vertices in their intersection, so  $|\mathcal{C}'| \geq 2$ .

Let  $C'$  and  $C''$  be two non-disjoint cliques in  $\mathcal{C}$  such that  $C' \in \mathcal{C}'$  and  $C'' \notin \mathcal{C}'$ . Let  $\mathcal{C}''$  denote  $\mathcal{C}' \cup \{C''\}$ . By the maximality of  $\mathcal{C}'$ , we have  $|\cap \mathcal{C}''| < \frac{1}{3}(\Delta + 1)$ . Suppose that  $\cap \mathcal{C}''$  is nonempty. Any vertex in  $\cap \mathcal{C}''$  is adjacent to the rest of  $\cup \mathcal{C}''$ , so  $|\cup \mathcal{C}''| \leq \Delta + 1$ . But this contradicts Lemma 3, so  $\cap \mathcal{C}''$  must indeed be empty.

Since  $C' \cap C'' \neq \emptyset$  it follows that  $|C' \cap C''| \geq \omega/2$ . On the other hand we also have  $|C' \setminus C''| \geq |\cap \mathcal{C}'| \geq \omega/2$  and so  $|C' \cap C''| = |\cap \mathcal{C}'| = \omega/2$ . Thus it is clear that the sets  $(C' \cap C'')$  and  $(\cap \mathcal{C}')$  partition  $C'$ . Also, no clique of  $\mathcal{C}'$  can intersect  $C'' \setminus C'$ , since a vertex in this intersection would be complete to  $C'$ , contradicting the fact that  $C'$  is a maximum clique. Further, no clique  $C$  of  $\mathcal{C}'$  other than  $C'$  can intersect  $C''$ , since this would imply that  $C$  and  $C''$  have nonempty intersection of size less than  $\frac{1}{3}(\Delta + 1)$ , which is impossible. Finally, it follows that there can be only one clique in  $\mathcal{C}' \setminus \{C'\}$ , otherwise  $|\cup \mathcal{C}'|$  would be greater than  $\Delta + 1$ .

We have shown that if  $|\cap \mathcal{C}| < \frac{1}{3}(\Delta + 1)$ , then no three cliques of  $\mathcal{C}$  intersect, and any two intersecting cliques intersect in exactly  $\frac{1}{3}(\Delta + 1)$  vertices. The first part of the lemma follows immediately from the fact that the strong product of a clique and a 3-cycle is itself a clique, thus  $G[\mathcal{C}]$  is either a path or a hole, and if it is a path it must have at least four vertices, since  $\cap \mathcal{C}$  is nonempty. The second part follows from the fact that if  $G$  contains the strong product of a clique and a hole as a subgraph  $G'$ , then  $G'$  is  $\Delta$ -regular and therefore since  $G$  is connected we have  $G' = G$ .  $\square$

### 3 Hitting the maximum cliques with a stable set

In order to find our desired stable set, we need the main intermediate result in the proof of Theorem 1.

**Theorem 5** (King [7]). *For a positive integer  $k$ , let  $G$  be a graph with vertices partitioned into cliques  $V_1, \dots, V_r$ . If for every  $i$  and every  $v \in V_i$ ,  $v$  has at most  $\min\{k, |V_i| - k\}$  neighbours outside  $V_i$ , then  $G$  contains a stable set of size  $r$ .*

*Proof of Theorem 2.* For fixed  $\omega(G) \geq 1$  we proceed by induction on  $|V(G)|$ ; the result trivially holds whenever  $|V(G)| \leq \omega(G)$ . Let  $\mathcal{C}$  be the set of maximum cliques in a graph  $G$ , and

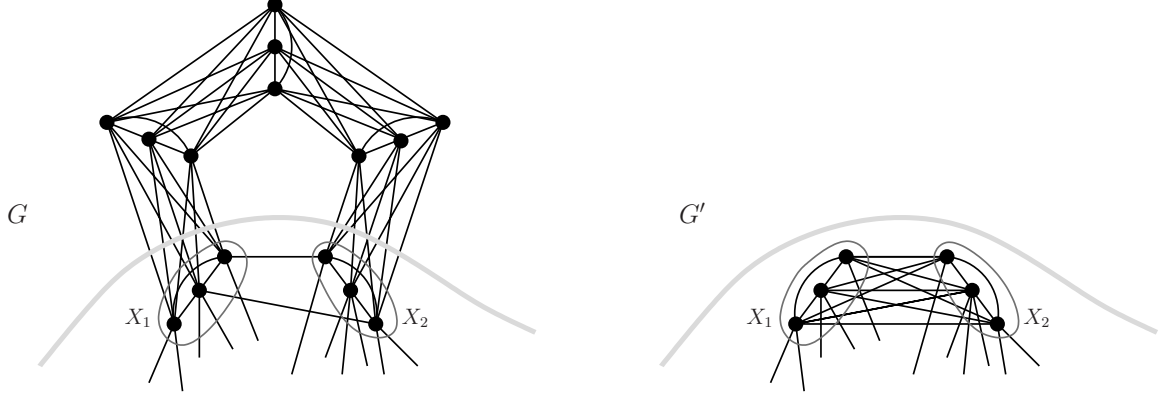


Figure 2: A reduction of a clique path for  $\ell = 5$

let  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$  be the partitioning of  $\mathcal{C}$  such that  $G[\mathcal{C}_1], G[\mathcal{C}_2], \dots, G[\mathcal{C}_k]$  are the connected components of the clique graph  $G[\mathcal{C}]$ . We consider two cases. The first case is basically the same as the proof of Theorem 1.

**Case 1:** For every  $1 \leq i \leq k$ ,  $\cap \mathcal{C}_i \neq \emptyset$ .

By Lemma 4, for every  $1 \leq i \leq k$  we have  $|\cap \mathcal{C}_i| \geq \frac{1}{3}(\Delta(G) + 1)$ . It suffices to show that there is a stable set in  $G$  intersecting each  $\cap \mathcal{C}_i$ . For a given  $i$ , every vertex in  $\cap \mathcal{C}_i$  has at most  $\Delta(G) + 1 - |\cup \mathcal{C}_i|$  neighbours in  $\cup_{j \neq i}(\cap \mathcal{C}_j)$ . Lemma 3 tells us that  $|\cup \mathcal{C}_i| + |\cap \mathcal{C}_i| \geq \frac{4}{3}(\Delta(G) + 1)$ . Therefore  $\Delta(G) + 1 - |\cup \mathcal{C}_i| \leq |\cap \mathcal{C}_i| - \frac{1}{3}(\Delta(G) + 1)$ . And since  $|\cup \mathcal{C}_i| \geq \omega(G) \geq \frac{2}{3}(\Delta(G) + 1)$ , we can see that a vertex in  $\cap \mathcal{C}_i$  has at most  $\min\{\frac{1}{3}(\Delta(G) + 1), |\cap \mathcal{C}_i| - \frac{1}{3}(\Delta(G) + 1)\}$  neighbours in  $\cup_{j \neq i}(\cap \mathcal{C}_j)$ . It therefore follows from Theorem 5 that there is a stable set in  $G$  intersecting each  $\cap \mathcal{C}_i$ . This completes Case 1.

**Case 2:** For some  $1 \leq i \leq k$ ,  $\cap \mathcal{C}_i = \emptyset$ .

Assume that  $\cap \mathcal{C}_1 = \emptyset$ . Lemma 4 tells us that either  $G$  is the strong product of  $K_{\omega(G)/2}$  and a hole, or  $G[\cup \mathcal{C}_i]$  contains as a subgraph the strong product of  $K_{\omega(G)/2}$  and a  $P_\ell$  for  $\ell \geq 4$ . In the former case the theorem clearly holds, so let us consider the latter case. If there is a vertex not in a clique of size  $\omega(G)$ , we can delete it and apply induction, so assume that no such vertex exists. Let the cliques of  $\mathcal{C}_1$  be  $C_1, \dots, C_{\ell-1}$  such that for  $1 \leq i \leq \ell - 2$ ,  $C_i$  and  $C_{i+1}$  intersect in exactly  $\omega(G)/2$  vertices. Let  $X_1$  denote  $C_1 \setminus C_2$  and let  $X_2$  denote  $C_{\ell-1} \setminus C_{\ell-2}$ .

We will construct a graph  $G'$  on fewer than  $|V(G)|$  vertices such that  $\omega(G') = \omega(G)$  and  $\Delta(G') \leq \Delta(G)$ , and apply induction to prove our result. To construct  $G'$  from  $G$  we delete  $\cup_{1 \leq i \leq \ell-2} (C_i \cap C_{i+1}) = (\cup \mathcal{C}_1) \setminus (X_1 \cup X_2)$  and add edges to make  $X_1 \cup X_2$  a clique of size  $\omega$  in  $G'$  (see Figure 2). Clearly  $G'$  has maximum degree at most  $\Delta(G)$ . We claim that  $G'$  has clique number  $\omega(G)$ . Suppose this is not the case. It follows that there exists a set  $Y_1 \subseteq X_1 \cup X_2$  and a set  $Y_2$  in  $V(G) \setminus \cup \mathcal{C}_1$  such that  $Y_1 \cup Y_2$  is a clique of size greater than  $\omega(G)$ . Let  $v$  be a vertex in  $Y_2$ . Since  $v$  is in an  $\omega(G)$ -clique in  $G$ , it has at most  $\omega(G)/2$

neighbours in  $X_1 \cup X_2$ , so  $|Y_1| \leq \omega(G)/2$ . Therefore  $|Y_2| > \omega(G)/2$ , which implies that some vertex in  $Y_1$  has at least  $\omega(G) - 1$  neighbours in  $\cup C_1$  and more than  $\omega(G)/2$  neighbours in  $Y_2$ , contradicting the fact that  $\omega(G) \geq \frac{2}{3}(\Delta(G) + 1)$ . Therefore  $G'$  has clique number  $\omega(G)$ .

By induction, there is a stable set  $S$  in  $G'$  hitting every  $\omega(G)$ -clique. Thus  $S$  is also a stable set in  $G$  intersecting  $X_1 \cup X_2$  exactly once. Without loss of generality let  $v$  be a vertex in  $X_1 \cap S$ . From  $S$  we will construct a stable set  $S'$  hitting every  $\omega(G)$ -clique in  $G$  in one of two ways, depending on the parity of  $\ell$ .

If  $\ell$  is even, let  $S'$  consist of  $S$  along with one vertex in  $C_{2k} \cap C_{2k+1}$  for each  $1 \leq k \leq (\ell/2) - 1$ . It is a routine exercise to confirm that  $S'$  is a stable set hitting every maximum clique in  $G$ .

If  $\ell$  is odd, let  $S'$  consist of  $S \setminus \{v\}$  along with  $C_{2k-1} \cap C_{2k}$  for each  $1 \leq k \leq (\ell - 1)/2$ . Again we can easily confirm that  $S'$  is a stable set hitting every maximum clique in  $G$ , because the only  $\omega(G)$ -clique intersecting  $C_1 \setminus C_2$  is  $C_1$ . This completes the proof.  $\square$

## 4 Hitting large maximal cliques with a stable set

Theorem 1 can be used to characterize minimum counterexamples to Reed's  $\chi$ ,  $\omega$ ,  $\Delta$  conjecture; see for example [1] §4. Motivated by the problem of similarly characterizing minimum counterexamples to the local strengthening of Reed's  $\chi$ ,  $\omega$ ,  $\Delta$  conjecture (see [4, 6]), King recently proposed the following unpublished conjecture:

**Conjecture 6.** *There exists a universal constant  $\epsilon > 0$  such that every graph contains a stable set hitting every maximal clique of size at least  $(1 - \epsilon)(\Delta + 1)$ .*

We conclude this note by disproving the conjecture.

**Theorem 7.** *For any  $\epsilon > 0$  there exists a graph in which every maximal clique has size at least  $(1 - \epsilon)(\Delta + 1)$ , and no stable set hits every maximal clique.*

*Proof.* Choose two positive integers  $k$  and  $t$  sufficiently large such that

$$(1 - \epsilon)(kt + 5t - 5) < kt + 2 - k. \tag{1}$$

We now construct a graph with vertices partitioned into sets  $A$  and  $B$  of size  $kt$  and  $5t$  respectively. We further partition  $A$  into  $A_1, \dots, A_t$  and  $B$  into  $B_1, \dots, B_t$  such that

1.  $A$  is a clique and each  $A_i$  has size  $k$
2. each  $B_i$  induces a 5-cycle, and there are no edges between  $B_i$  and  $B_j$  for  $i \neq j$
3. vertices  $u \in A_i$  and  $v \in B_j$  are adjacent precisely when  $i \neq j$ .

Thus we can see that the unique maximum clique in  $G$  is  $\cup_i A_i$ , with size  $kt$ . All other maximal cliques of  $G$  consist of two vertices in  $B$  and  $k(t - 1)$  vertices of  $A$ . The maximum degree of the graph is  $kt + 5t - 6$ , achieved by all vertices in  $A$ . By (1), every maximal clique has size greater than  $(1 - \epsilon)(\Delta + 1)$ .

It therefore suffices to prove that no stable set intersects every maximal clique. Suppose we have a stable set  $S$  intersecting every maximal clique. Since  $A$  is a maximal clique, without loss of generality we can assume  $S$  intersects  $A_1$ , and therefore  $S \setminus A_1 \subseteq B_1$ . But then there must remain two adjacent vertices in  $B_1 \setminus S$ . Together with  $\cup_{j \neq 1} A_j$  these vertices form a maximal clique in  $G$ . This contradiction completes the proof.  $\square$

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