THE KRAUSZ DIMENSION OF A GRAPH AFTER THE REMOVAL OF AN EDGE

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Abstract. A Krausz partition of a graph is a partition of the edges of the graph into complete subgraphs. The Krausz dimension of a graph is the smallest integer k such that a graph has a Krausz partition in which every vertex belongs to at most k of the complete graphs of the partition. In [2], Beineke and Broere observed that the Krausz dimension of a graph might increase after the removal of an edge and conjectured that for graphs of a given order the greatest change happens when we remove an edge from the complete graph. In this note we provide a short proof of their conjecture.

1. Introduction

In [3] Krausz proved that a graph is a line graph if and only if its edges can be partitioned into complete subgraphs such that every vertex of the graph belongs to at most two of them. Motivated by this result, Beineke and Broere [1] generalized the notion of line graphs by allowing the vertices to belong to more parts.

A *Krausz partition* of a graph is defined to be any partition of the edges of the graph into complete subgraphs. The *Krausz dimension* of a graph G, denoted by $\dim(G)$, is then defined to be the smallest integer k such that G has a Krausz partition in which every vertex of G appears in at most k of the parts. Thus a graph has Krausz dimension one if and only if it is complete and it has Krausz dimension at most two if and only if it is a line graph.

In [1, 2], Beineke and Broere investigated several properties of the Krausz dimension of a graph. In [2] they observed that the Krausz dimension of a graph can increase after the removal of an edge. Moreover they proved that $\dim(K_n^-) \geqslant \lfloor \frac{3+\sqrt{4n+1}}{2} \rfloor$ whenever $n \geqslant 7$, where K_n^- denotes the graph obtained from K_n after the removal of an edge. Thus the increase in the Krausz dimension can be arbitrarily large. In the same paper, the authors asked whether amongst all graphs of a given order the complete graph exhibits the greatest increase in dimension after the removal of an edge. We will prove in the next section that this is indeed the case.

What about the greatest increase in dimension amongst all graphs of the same size? For a positive integer m, let $\ell(m)$ be the largest integer such that $\binom{\ell(m)}{2} \leqslant m$ and let $G(m)$ be the graph consisting of a clique of size $\ell(m)$ together with $m - \binom{\ell(m)}{2}$ independent edges. We will show that amongst all graphs of size m, the graph $G(m)$ exhibits the greatest increase in dimension after removal of an edge.

One may wonder whether the graphs K_n and $G(m)$ are the unique graphs (on n vertices and on m edges respectively) which exhibit the greatest increase in dimension after the removal of an edge. $G(m)$ is obviously not unique as adding isolated vertices to it leads to a new extremal graph so it only makes sense to consider whether K_n is the unique

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graph on n vertices which exhibits the greatest increase in dimension after the removal of an edge. We will explain why this is in general not the case and give a criterion for uniqueness. This criterion will depend on the values of $\dim(K_n^-)$ which are unfortunately not known.

2. Proofs of the results

Let G be a graph of order n and Krausz dimension k. Let $\{H_1, \dots, H_r\}$ be a Krausz partition of G such that every vertex of G belongs to at most k parts. Let e be an edge of G and assume that it belongs to H_r. Suppose H_r has order ℓ and let $\{F_1, \dots F_s\}$ be a Krausz partition of $H_r - e$ such that every vertex of $H_r - e$ belongs to at most $\dim(K_\ell^-)$ parts. Then $\{H_1, \dots, H_{r-1}, F_1, \dots, F_s\}$ is a Krausz partition of $G - e$. Moreover, every vertex of $G - e$ which does not belong to H_r belongs to at most k parts while every vertex of $G - e$ which belongs to H_r belongs to at most $k - 1 + \dim(K_\ell^-)$ parts. It follows that after removing the edge e the Krausz dimension of G has increased by at most $\dim(K_\ell^-) - 1$. Since removing a vertex from a graph obviously does not increase the Krausz dimension, we deduce that after removing e the Krausz dimension of G has increased by at most $\dim(K_\ell^-) - 1 \leqslant \dim(K_n^-) - 1 = \dim(K_n^-) - \dim(K_n)$.

A similar proof works when G is a graph of size m. The only change is in the conclusion where we first observe that $\ell \leqslant \ell(m)$ and so $\dim(K_\ell^-) - 1 \leqslant \dim(K_{\ell(m)}^-) - 1 = \dim(G(m)^-) - \dim(G(m))$, where $G(m)^-$ denotes the graph obtained from $G(m)$ by removing an edge from its clique of maximum size.

We now move to the question of whether K_n is the unique graph on n vertices which exhibits the largest increase in the Krausz dimension after the removal of an edge. In [2] it is proved that $\dim(K_n^-) = (1 + o(1))\sqrt{n}$. In particular, it follows that there are many values of n for which $\dim(K_{n-1}^-) = \dim(K_n^-)$. But for each such value of n, a clique on $n-1$ vertices together with an isolated vertex exhibits the same change in the dimension after the removal of an edge as K_n does and so the extremal graph is not unique. On the other hand we claim that if $\dim(K_{n-1}^-) < \dim(K_n^-)$ then K_n is the unique graph on n vertices which exhibits the greatest increase in dimension after the removal of an edge. Indeed looking back at the proof that K_n is extremal, we see that for any other graph on n vertices if we remove an edge we increase the Krausz dimension by at most $\dim(K_\ell^-) - 1$ for some $\ell \leqslant n - 1$. In particular the dimension has increased by at most $\dim(K_{n-1}^-) - 1 < \dim(K_n^-) - \dim(K_n)$.

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