A proof of the dense version of Lovász conjecture

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Abstract

We prove that every sufficiently large dense connected vertex-transitive graph is Hamiltonian.

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1 Introduction

The decision problems of whether a graph contains a Hamilton cycle or a Hamilton path are two of the most famous NP-complete problems, and so it is unlikely that there exist good characterizations of such graphs. For this reason, it is natural to ask for sufficient conditions which ensure the existence of a Hamilton cycle or a Hamilton path. To this direction, the following well-known conjecture of Lovász [5] is still wide open.

Conjecture 1.1 *Every connected vertex-transitive graph has a Hamilton path.*

At the moment no counterexample is known. Moreover, there are only five known examples of connected vertex-transitive graphs having no Hamilton cycle. These are K_2, the Petersen graph, the Coxeter graph and the graphs obtained from the Petersen and Coxeter graphs by replacing every vertex with a triangle.

The conjecture has attracted a lot of interest from researchers and there is no common agreement as to its validity. We will omit any overview of the vast research these questions have motivated, referring the reader to the following surveys [8,2,4,7] and their references.

In [1] we proved that every sufficiently large dense connected vertex-transitive graph is Hamiltonian.

Theorem 1.2 *For every $\alpha > 0$ there exists an n_0 such that every connected vertex-transitive graph on $n \geqslant n_0$ vertices of valency at least αn contains a Hamilton cycle.*

Our aim here is to give an outline of the proof of the above theorem.

2 Outline of the proof

2.1 Almost perfect matchings in the reduced graph

The proof of the above result uses Szemerédi's Regularity Lemma. The first step after applying the Regularity Lemma is to obtain an almost perfect matching in the reduced graph R. Indeed, matchings in the cluster graph are a crucial auxiliary structure for proving existence of a long cycle in G as

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was already observed by Łuczak [6]. The way we employ Łuczak's technique is detailed in Subsection 2.2.

It is well known (see e.g. [3, Theorem 3.5.1]) that every vertex-transitive graph contains a matching covering all but at most one of its vertices. Unfortunately, the vertex-transitiveness need not be inherited by R. To explain how to find the almost perfect matching in R we first recall some notions from matching theory.

Given a graph G we recall that its fractional covering number $\tau^*(G)$ is given by the following optimization problem: We want to give a non-negative weight w_x to each vertex x of G such that $w_x + w_y \geqslant 1$ whenever xy is an edge of G and such that the sum of the weights is as small as possible. In a similar manner the fractional matching number $\nu^*(G)$ is given by the following optimization problem: We want to give a non-negative weight w_e to each vertex e of G such that at each vertex the sum of the weights of the incident edges is at most 1 and such that the total sum of the weights is as large as possible.

It is straightforward that $\nu^*(G) \leqslant \tau^*(G)$. In fact, the duality of linear programming guarantees that we have equality. To find the almost perfect matching in R we proceed as follows: It is easy to show that every (non-trivial) vertex-transitive graph G on n vertices satisfies $\tau^*(G) = n/2$ and therefore $\nu^*(G) = n/2$. Moreover, it can be shown that this number does not change by much after the removal of a small proportion of its edges. One can then show that the property of having a large fractional matching is inherited by R. The final ingredient is the half-integrality property of fractional matchings: In a fractional matching of maximum weight we may assume that every edge gets a weight in the set $\{0, 1/2, 1\}$. By a standard procedure of splitting every cluster of R into two pieces we may pass into a new reduced graph R' with an almost perfect matching M. By moving all clusters not incident to M into the exceptional set we can in fact assume that M is a perfect matching in R'.

2.2 *Strategy for finding the Hamilton cycle*

To find the Hamilton cycle we now proceed in three steps:

(i) Make the pairs of clusters corresponding to the edges of M super-regular by moving vertices to the exceptional set.

(ii) Find a cycle C in G containing all vertices of the exceptional set such that in addition for every matching edge XY we have $|V(C) \cap X| = |V(C) \cap Y|$.

(iii) Use the Blow-up Lemma within each matching edge to turn C into a Hamilton cycle.

We will avoid mentioning any of the technicalities that arise in executing the above steps and refer the reader to [1] for full details of the proof. We do mention however two important aspects that arise and explain how these are treated.

2.3 Iron-Connectivity

We would like to know that R is connected. In fact, in order to execute step (ii) above we would like to know that R has high vertex connectivity, for example to be $(\gamma|R|)$-connected for some constant γ.

It is well-known (see e.g. [3, Theorem 3.4.2]) that a dense vertex-transitive graph has high vertex connectivity. To be more precise, every vertex-transitive graph of degree k is $2(k+1)/3$-connected. Unfortunately, this property is not inherited by R. For example if G is the union of two cliques A, B on $n/2$ vertices each together with a perfect matching between A and B then G is $n/2$-connected. However, if we apply the Regularity Lemma with prepartition A, B, then the reduced graph will not even be connected as the cross-edges are too sparse. In the proof of our main result we introduce two new notions of connectivity: robust and iron connectivity. The important notion is the second while the first notion is more for convenience.

We say that a graph G is ℓ-robust if G remains connected even after removal of an arbitrary set $E' \subseteq E(G)$ with $\Delta(E') \leqslant \ell$. (The last condition denotes that we have removed at most ℓ edges incident to every vertex.) We say that G is ℓ-iron if G stays connected after simultaneous removal of an arbitrary edge-set $E' \subseteq E(G)$ with $\Delta(E') \leqslant \ell$ and an arbitrary vertex-set $U \subseteq V$ with $|U| \leqslant \ell$.

Recall that when we apply the Regularity Lemma we often set aside some set of vertices and some set of edges incident to every vertex that we 'cannot see' just by looking at the reduced graph R. To be more precise, applying the degree form of the Regularity Lemma with parameters ε and d discards an exceptional set V_0 of size and most εn and an edge set E such that every vertex is incident to at most $(d+\varepsilon)n$ edges of E. The latter is needed because we only want to work with pairs of clusters which have density at least d, the others being too sparse to work with. So if G is $(d+\varepsilon)n$-iron, then we can ensure that even after the removal of these vertices and edges what remains is still connected. In fact, it is not too difficult to show that if a graph has high iron-connectivity then the reduced graph also has high iron-connectivity. For this reason, the iron-connectivity seems very well-suited to use in combination with the Regularity Lemma.

Of course a dense vertex-transitive graph might be very far from being iron-connected. We can show however that we can partition any dense vertex-transitive graph into a bounded number of highly iron-connected pieces. Furthermore, all of these pieces can be taken to be isomorphic. The main result in this direction is the following.

Lemma 2.1 *For every $\alpha > 0$ there exist $\beta, R, N_0 > 0$ such that the following holds: Suppose G is a vertex-transitive graph of order $n > N_0$ and valency at least αn. Then there exists a partition $V(G) = V_1 \cup \dots \cup V_r$ into $r < R$ parts such that all the graphs $G[V_i]$ are isomorphic to a graph G' which is vertex-transitive and (βn)-iron. Furthermore, for each $g \in \mathrm{Aut}(G)$ and each $1 \leqslant j \leqslant r$ we have $g(V_j) \in \{V_1, \dots, V_r\}$.*

So in the proof of the main theorem we can concentrate on finding Hamilton cycles in each of these pieces separately (we only need to find a Hamilton cycle in one of them as they are isomorphic) and then glueing these Hamilton cycles together.

2.4 Bipartiteness

The final issue we will discuss in this abstract is about bipartiteness. There would be some complications arising in the proof if the graph G was not bipartite but was very 'close' to being bipartite. Fortunately in this case we can say a lot about the structure of the graph which enables the proof to go through.

Let us say that an n-vertex graph is ε-close to bipartite if we can remove at most εn^2 edges to make it bipartite. The main structural result we use is the following.

Lemma 2.2 *Let $0 < c < 1/17$ be arbitrary. Suppose that G is a cn-iron vertex-transitive graph G on n vertices which is c^4-close to bipartiteness. Then there exists a bipartition $V(G) = A \cup B$ such that $|A| = |B|$, for each $u \in A$ and each $v \in B$ we have $\deg(u, A) \leqslant 6c^2 n$, and $\deg(v, B) \leqslant 6c^2 n$. Furthermore, we have $g(A) = A$ or $g(A) = B$ for each $g \in \mathrm{Aut}(G)$.*

This enables us to split the proof into two parts according to whether the graph (or rather the iron-connected pieces of the graph we obtain from Lemma 2.1) is c^4-close to bipartite or not. The case where we are far from bipartiteness is significantly easier but even when we are close to bipartiteness we can still find the Hamilton cycle using the information provided in the above result.

References

[1] Christofides D., J. Hladký and A. Máthé, *Hamilton cycles in dense vertex transitive graphs*, preprint, `arXiv:1008.2193`.

[2] Curran S. J., and J. A. Gallian, *Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey*, Discrete Math. **156** (1996), 1–18.

[3] Godsil C., and G. Royle, "Algebraic graph theory," Springer, New York, 2001

[4] Kutnar K., and D. Marušič, *Hamilton cycles and paths in vertex-transitive graphs—current directions*, Discrete Math. **309** (2009), 5491–5500.

[5] Lovász L., *Problem 11* in "Combinatorial Structures and their Applications" Gordon and Breach Science Publishers, New York, 1970

[6] Łuczak T., $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J. Combin. Theory Ser. B **75** (1999), 174–187.

[7] Pak I., and R. Radoičić, *Hamiltonian paths in Cayley graphs*, Discrete Math. **309** (2009), 5501–5508.

[8] Witte D. and J. A. Gallian, *A survey: Hamiltonian cycles in Cayley graphs*, Discrete Math. **51** (1984), 293–304.